A Qualitative Approach to Sketch the Graph of a Function

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Introduction

One common method for sketching the graph of a function includes the definition of the domain, study of zeros of the first and second derivatives, calculation of limits at some critical points, etc. All this information is assembled (sometimes in a tableau) and then translated to the Cartesian plane.

Obtaining all the information mentioned above requires considerable algebraic skill. Many of our students did not have these capabilities. Also these methods provide accuracy (for example, the exact location of extreme points) which is sometimes unnecessary if the goal is only to have a general idea of what the graph looks like. The result was that most students did not learn very much about how to sketch the graph of a function.

Although the method referred to in this manuscript is a general procedure widely used in calculus books, there are other ways of sketching the graph of a function. For example, when the graph of $\sin(x + 3)$ is required, no reference is made to the domain, derivative, etc. One simply thinks that it is obtained by moving the graph of $\sin(x)$ three units to the left.

This second way of sketching the graph is more qualitative and global. One always considers how the desired graph is related to another well-known graph. For instance, the graph of $\sin(x + 3)$ is the graph of $\sin(x)$ moved three units to the left.

This second way of thinking about graphs involves mainly functions of the form $af(x+b)$, $|f(x)|$, $f(|x|)$, where the shape of graph of f is known. Because these kind of functions are frequently used in elementary science courses, it is also usually presented in most calculus courses.

Students need to know that the graph of $f(x + a)$ has the same shape as the graph of $f(x)$, but the position is different and depends on a. The usual arguments that the height of $f(x + a)$ at x is the same as the height of $f(x)$ at $x + a$ did not work.

Students could tell if a formula was of the kind $f(x + a)$ or $a\,f(x)$ or $-f(x)$, etc. They can also move the graph of f to obtain the graph of $f(x + a)$ was the graph of f translated horizontally $-a$ units. To help the students understand this approach, geometric methods were developed that would provide alternative models.

The concept of paths (Alson, 1989) was a first step in this direction. The use of paths offered a quick way of graphing functions of the kind $f(x + a)$, $af(x)$, and $a + f(x)$. As a result, students felt that the graph obtained was really the one assigned.

For example, to convince the student of the relationship between the graphs of $f(x + a)$ and $f(x)$ the pupil was asked to draw the graphs of a set of formulae of the form $f(x + a)$ using paths. After the graphs were completed, the student had to discover, by analyzing his or her results, the relationship between the graph of $f(x)$, the graph of $f(x + a)$, and $-a$. The fact that the application of paths to graphing $f(x + a)$ requires the graphing of $f(x)$ is an advantage.

There is widespread confusion about functions such as $\sin(ax + b)$. Many people think that to graph them you have to shrink a times (if $a > 0$) the sine graph and then move horizontally $-b$ spaces. Others think that the order in which the geometrical operations are done does not matter. Checking at $x = 0$ and $x = 1$ shows that the right order is: first move the graph of $\sin(x)$ horizontally $-b$ spaces and then shrink the graph a times. Nevertheless, it was not clear why this was so. Trying to understand this point led to the definition of two kinds of graphical operators.

The second step in the building of geometrical methods involved these graphical operators. The content of this article is the description of such operators and their use in providing a global way of sketching the graphs of composite functions. This method is the natural generalization of the second way of thinking about graphs of functions described above. The section on Heuristic Remarks is intended to show how most of the ideas about shapes of graphs are organized using graphical operators in an implicit way. This article also explains how the method has been taught to students and some comments about the results obtained. Finally, some general remarks are contained in the Conclusions Section.

Definition of Graphic Operators

For any 1-1 function f, there are two operators which can be associated with f:

$$T_f^V(x, y) = (x, f(y))$$

and

$$T_f^H(x, y) = (f^{-1}(x), y).$$

If $f^{-1}(x)$ is understood as the set $\{z \mid f(z) = x\}$, the last operator is automatically extended to functions which are not 1-1.

Notice that the first operator induces a movement of points along the y axis and the second operator induces a movement of points along the x axis. This is why they are referred to as vertical and horizontal operators, respectively. Notice that the superscripts H and V will remind us, respectively, of the horizontal movement and the vertical movement induced by the operators.

The importance of both operators comes from the following equalities:

$$\{ (x, y) \mid y = f(g(x)) \} = \{ (x, y) \mid (x, y) = T_f^V(x, g(x)) \}$$

and

$$\{ (x, y) \mid y = f(g(x)) \} = \{ (x, y) \mid (x, y) = T_g^H(g(x), f(g(x))) \}$$

These equalities can be rewritten as:

$$\text{Graph}(fog) = T_f^V(\text{Graph}(g)) \qquad (1)$$

and

$$\text{Graph}(fog) = T_g^H(\text{Graph}(f)) \qquad (2)$$

Thus the graph of fog may be obtained in two ways: (a) modifying the graph of g using the vertical operator induced by f or (b) modifying the graph of f using the horizontal operator induced by g.

To illustrate the use of these properties, an example will be used. Let $(fog)(x) = \sin(x + 3)$. Now applying property (2), it is clearly understood that the graph of $\sin(x + 3)$ is obtained by applying the operator T^H_{x+3} to the graph of $\sin(x)$. Since the inverse of $x + 3$ is $x - 3$, $T^H_{x+3}(x, y) = (g^{-1}(x), y) = (x - 3, y)$, i.e., T^H_{x+3} is a horizontal translation three units to the left. Therefore, the shape of the graph of $\sin(x + 3)$ must be the same as the shape of the graph of $\sin(x)$ but its position (referred to the x axis) is three units to the left.

From the example, it is clear that the operators will be useless for sketching of graph purposes unless a good geometric description of them is available. The example works well because the transformation defined by $T(x, y) = (x - 3, y)$ has a well known geometric description--it is a horizontal translation.

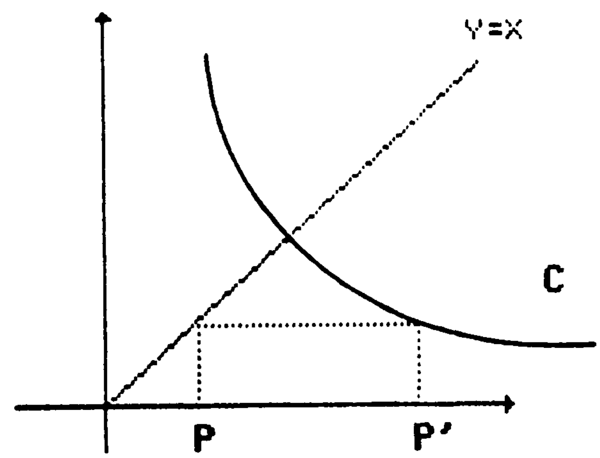
The next section shows the way in which the students were taught to obtain a geometric description for the operators of elementary functions like $1/x$, $|x|$, $\ln x$, $\sin(x)$, etc.

The Discovery of the Geometric Description of the Operators

If P is a point of the x-axis and C is the graph of the function $1/x$, a new point P' can be obtained following this procedure. Go vertically from P to the line $y = x$. Turn (to the left or to the right) 90 degrees and go horizontally until the curve C is reached. Finally, return to the x axis vertically. Figure 1 shows the procedure when it is applied for $x > 0$.

If the procedure just described is applied to some points of the x axis and the curve C of Figure 1, the following properties will soon become evident:

Figure 1.



(i) Points from the interval $(0, 1)$ are transformed into points of the interval $(1, \infty)$.

(ii) Points from the interval $(1, \infty)$ are transformed into points of the interval $(0, 1)$.

(iii) 1 is a fixed point.

(iv) The closer P is to zero (or 1) the closer P' is to ∞ (or 1), i.e., there is an inversion around 1.

(v) The image of the point 0 is the empty set, i.e., zero disappears.

Properties (i), (ii), (iii), (iv), and (v) conform to a global qualitative description of the process. Similar properties might be obtained for $x < 0$. They will be considered the geometric description of the operator $T^H_{1/x}$ (the horizontal operator of $1/x$).

If you apply, these geometric movements to the graph of $\sin(x)$ (for $x > 0$), all the oscillations of the graph of $\sin(x)$ (for $x > 1$) will be inverted around 1 and compacted into the interval $(0, 1)$ (Properties (iv) and (ii)) and the graph of sine for $0 < x < 1$ will be inverted around 1 and then stretched (horizontally) until it covers the part corresponding to the interval $(1, \infty)$ (Properties (iv) and (i)). The height of the curve at $x = 1$ does not change (Property (iii)). A similar argument can be applied to the graph of $\sin(x)$ for $x < 0$. Finally, the domain of the new curve does not contain zero because the operator does not transform any point of the x axis to zero. This description is illustrated in Figure 2, for $x > 0$.

So far, some graphical properties of the horizontal operator $1/x$ by means of path exploration have been determined. Also, its application was discussed in an example. The description of the vertical operator may be obtained in a similar way. Instead of starting from the x axis, start from the y axis. After reaching the line $y = x$, look for the curve vertically, and after reaching it, return to the y axis. Figure 3 illustrates the process.

To help a student verbalize the geometric description of the process the teacher can ask him or her a few questions such as: Has zero any image by the process? If $0 < y < 1$, where is the image of y? Does the process have any fixed points?

Figure 2.

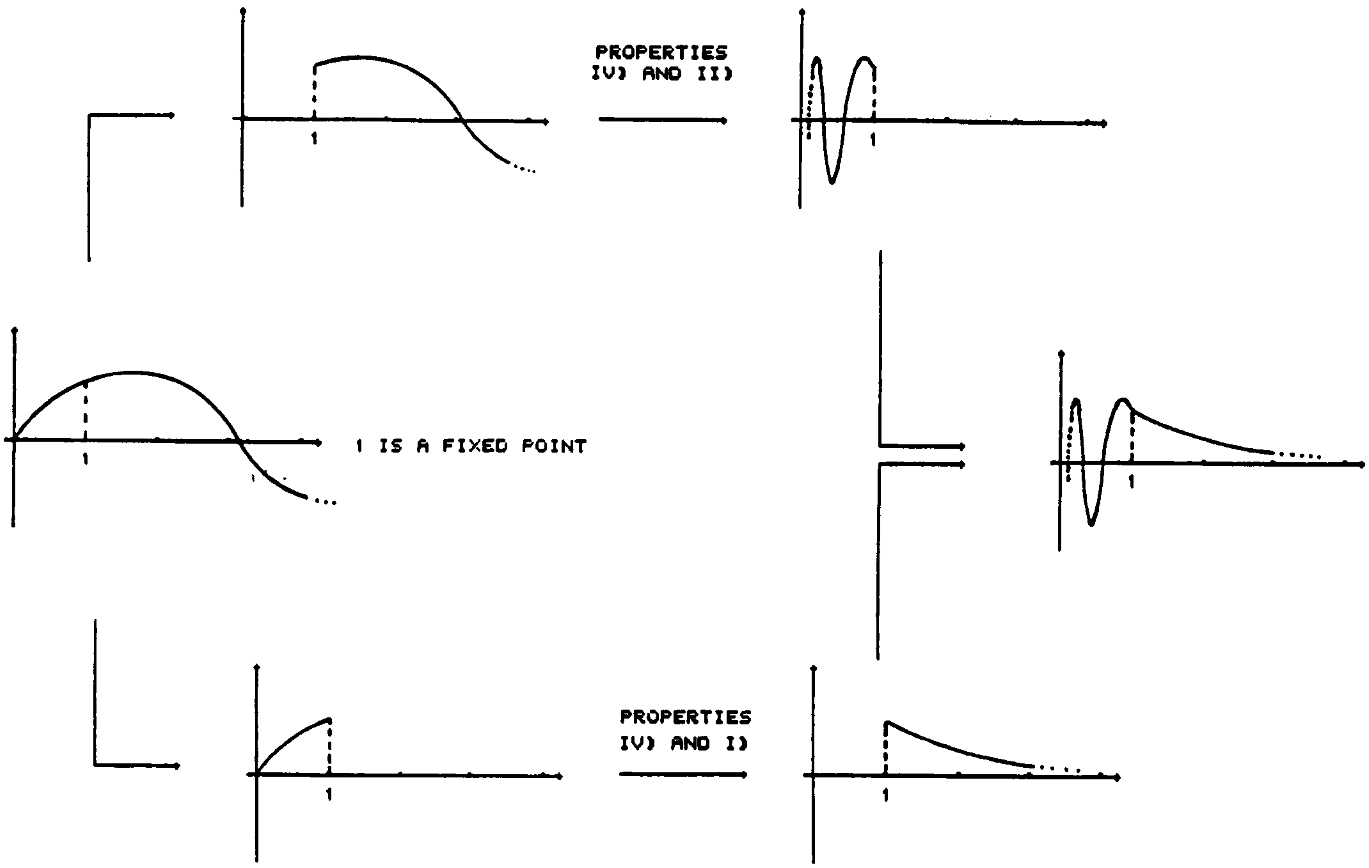
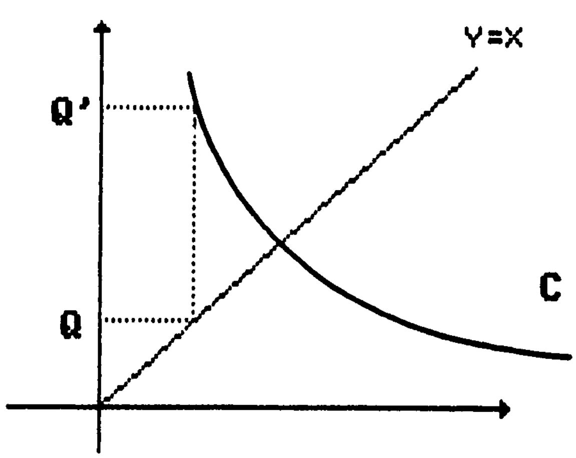


Figure 3.



This path process shown for the operators of $1/x$ might be applied to any function f; it is enough to draw the curve of f, instead of the one of $1/x$. In each case, after the student has worked with some points, the teacher should help the student verbalize the geometric description.

The next section shows that most of the usual heuristic commentaries about the shape of the graph of a composite function implicitly take into account the operators that have been defined.

Heuristic Remarks About the Shape of a Graph

Example 1

In standard courses, the shape of the graph of sin $(1/x)$ is not studied using limits, derivatives, zeros of the derivative, etc., because it makes the explanation too involved. Frequently, the following kind of explanation is preferred, "If x is near zero, small variations in x will produce big variations in $1/x$. In the case of sin ($1/x$) the interval ($1/[(2\pi)n]$, $1/[(2\pi)(n+1)]$) is transformed by $1/x$ into the interval ($(2\pi)n$, $(2\pi)(n+1)$) and the graph of sine on this interval is a whole period or cycle. Therefore, the graph of sin ($1/x$) on the interval ($1/[(2\pi)n]$, $1/[(2\pi)(n+1)]$] will consist of a cycle. Thus, there will be a lot of oscillation near zero. If x is very big . . . "

Students with a poor mathematical background have trouble following this reasoning with magnitudes and translating it to a lot of oscillation on the graph. The shape of sin ($1/x$) is explained looking at the effect of sin (x) on the magnitude $1/x$

(which is the height of the graph $1/x$ at x). Using the definitions introduced under Definition of Graphic Operators, it can be recognized that the above explanation uses $T^V_{\sin}(G(1/x))$ in an implicit way. Under The Discovery of the Geometric Description of the Operators, it was shown that a clearer explanation is obtained when $T^H_{1/x}(G(\sin))$ is used. (Here $G(f)$ means graph of f.)

Example 2

"The graph of $\sin(x+3)$ is obtained by translating the graph of $\sin(x)$ three spaces to the left."

The reasoning is done using $T^H_{x+3}(G(\sin)) = G(\sin(x+3))$.

Example 3

"The graph of $\cos(2x)$ is obtained from the graph of $\cos(x)$ by shrinking it twice along the x axis towards zero."

In this case, $T^H_{2x}(G(\cos)) = G(\cos(2x))$ is applied.

Example 4

"The graph of $(\sin(x))^2$ will look like the one of sine where $\sin(x) > 0$, but when $\sin(x) < 0$ . . . ."

Here the graph of $\sin(x)$ is modified using $T^V_{x^2}$. $T^V_{x^2}(G(\sin)) = G((\sin(x))^2)$ is used.

Example 5

When the graph of $1/(x^2 - 2)$ is studied, the first step is to calculate the domain of the function. Then the limits of the function at $\sqrt{2}, -\sqrt{2}, -\infty$ and ∞ are calculated. The critical points are found using derivatives. Finally, all this information is translated into the Cartesian plane to build up the graph of the function. In this example, $T^V_{1/x}(G(x^2 - 2))$ is used. Under General Methods, it will be shown how to build the graph of $1/(x^2 - 2)$ using $T^H_{x^2} T^H_{x-2}(G(1/x)) = G(1/(x^2-2))$.

Until now, rough graphing of the composition of two elementary functions has been discussed. The fact that the operators are not used explicitly has prevented their use with compositions of more than two elementary functions. The next section generalizes the results and shows how they can be applied to this kind of function.

General Methods

The generalization of the method using graphical operators rests upon the following two properties (one for each operator).

$$T^H_{f_1 o f_2 o \ldots o f_n}(G(g)) = T^H_{f_n}(\ldots\ldots(T^H_{f_1}(G(g))\ldots\ldots). \quad (3)$$

and

$$T^V_{f_1 o f_2 o \ldots o f_n}(G(g)) = T^V_{f_1}(\ldots\ldots(T^V_{f_n}(G(g))\ldots\ldots). \quad (4)$$

To avoid technicalities, include only the proof of the first property (which is the least evident) when $n = 2$ and the fi are 1-1 functions.

$$T^H_{f_1 o f_2}(x, y) = ((f_1 o f_2)^{-1}(x), y) = ((f_2^{-1} o f_1^{-1})(x), y) = T^H_{f_2} T^H_{f_1}(x, y)$$

An example will be used to illustrate how to graph when only horizontal operators are used. To graph $f(x) = 1/(x^2 - 2)$, it is necessary to see the function f as a composite of functions fi for which T^H_{fi} is known (that is $f = f_3 o f_2 o f_1$ and $f_1 = x^2$, $f_2 = x - 2$, $f_3 = 1/x$). If property (3) is applied, the first graph to work on is the graph of $1/x$. Then a new graph is obtained using T^H_{x-2}, which is the graph of $1/(x-2)$ (the operator T^H_{x-2} translates 2 units to the right) (see Figure 4). The next (and last) step is to apply the operator $T^H_{x^2}$ to the graph of $1/(x-2)$.

Using the procedure of Figure 1, the following geometrical description of $T^H_{x^2}$ is obtained:

(a) The part of the graph corresponding to $x < 0$ is eliminated.

(b) 0 and 1 are fixed points.

(c) Movement of points (shrinkage) towards 1.

(d) A point of the positive part of the x axis is transformed into two symmetric points (the y axis is the axis of symmetry).

As a consequence of the shrinkage (Property c), the asymptote $x = 2$ of the graph of $1/(x-2)$ will be transformed into the asymptote $x = \sqrt{2}$ because the inverse function of x^2, $x > 0$ is $\sqrt{x}$ (see Figure 4). The other asymptote of the graph of $f(x) = 1/(x^2-2)$ appears when Property (d) of the operator $T^H_{x^2}$ is applied.

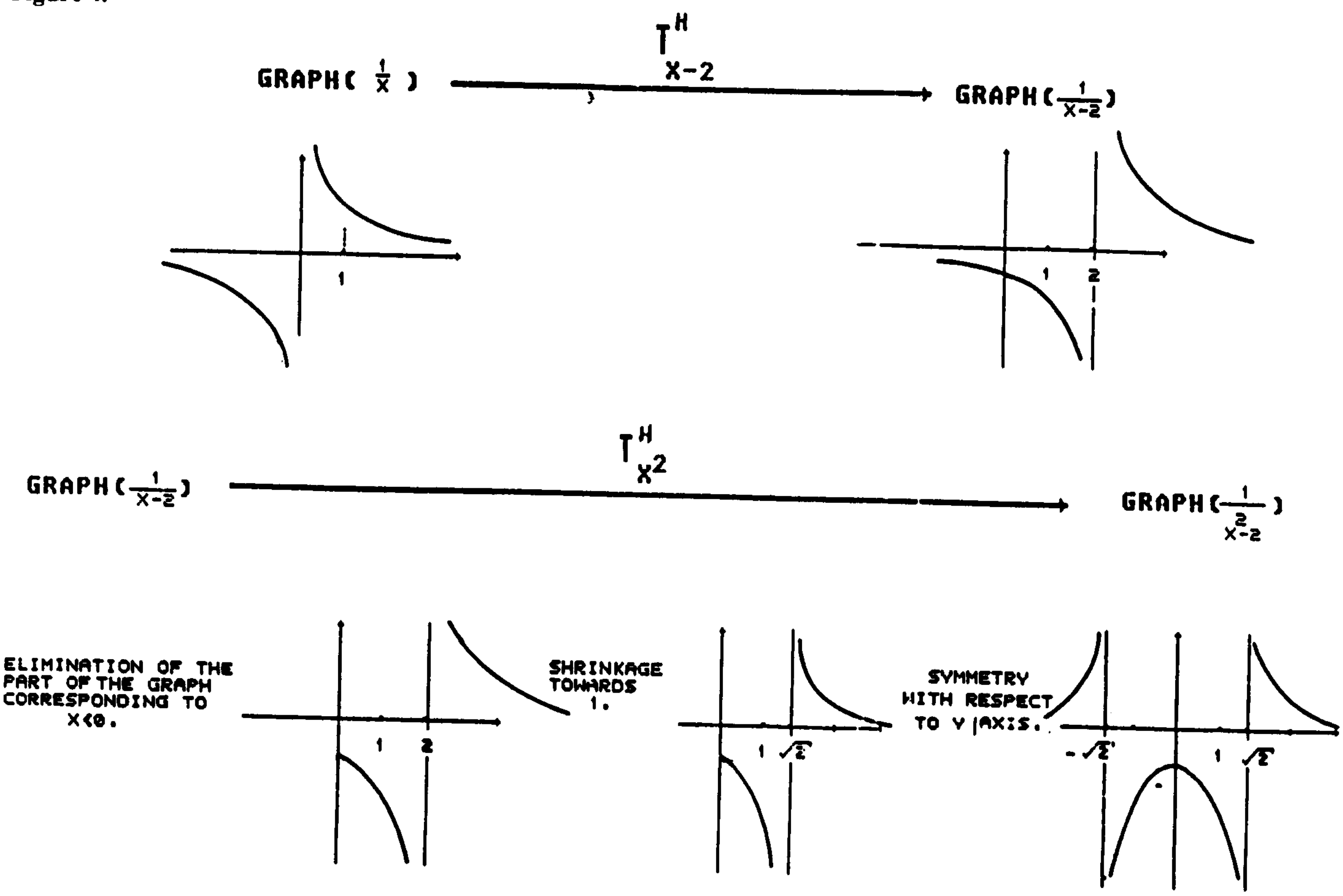
Notice, that here again as in the example of $\sin(1/x)$, a non-usual way of sketching the graph of $1/(x^2-2)$ is used.

It was pointed out that to graph $G(gof)$ we have two choices: $T^H_f(G(g))$ or $T^V_g(G(f))$. The choice will be determined by the difficulty of each way. For instance, to graph the absolute value of f it is better to use $T^V_{||}(G(f))$. When the function to graph is the composition of more than two elementary functions, there are several possible ways of using the operators. Sometimes a combination of two different kinds of operators is better. For example, to graph $c\sin(ax+b)+d$ using $T^V_{x+d}(T^V_{cx}(T^H_{ax}(T^H_{x+b}(G(\sin)))))$ is the best way.

Some Teaching Experience

Before teaching students about the use of these operators, (mainly those of the horizontal kind) and the graphing method derived from it, they were taught how to decompose functions. To avoid the use of formal language like the word operator, special tutorial material was prepared (Alson, 1987). The path mechanism shown under The Discovery of the Geometric Description of the Operators was used by the students to

Figure 4.



discover the global qualitative properties used for the geometric description of the operators. The teacher helped to work out the geometric description of the operators for each of the elementary functions.

Although the easiest way to graph is often achieved by using a combination of horizontal and vertical operators, the use of horizontal operators were emphasized. This was due to a lack of knowledge about what student response would be. After only two weeks, most of the students were able to graph functions like $\sqrt{\sin(x+2)}$ and $3\cos(1-x)+2$.

Experience tends to confirm that computer science students can use both kinds of operators. Students with a poor mathematical background have difficulty understanding and applying operators like $T^H_{\sin}$. Nevertheless, the operators T^H_{ax}, T^H_{x+b}, T^H_{ax}, T^V_{x+b}, $T^H_{||}$, and $T^V_{||}$ are generally comprehended by most of the students.

The mistakes that do occur are of two kinds: (a) the big ones which come from a wrong application of properties (1) and (2) (for example graphing $af(x)$ instead of $f(ax)$ because $T^V_{ax}(G(f))$ is used instead of $T^H_{ax}(G(f))$ and (b) the small mistakes which come from improper application of the correct operators.

Conclusions

These results show that students are able to graph a much larger class of functions than those graphed in usual calculus courses. The usual approach seems to have been limited for two reasons: (a) the non-explicit use of the operators and (b) the fact that the operators normally used were only those which could be related to well-known geometric transformations such as translations, symmetries, etc. (Less precise ideas like stretch or compress were not used.)

The use of operators provides a powerful tool for the sketching of graphs of functions. This technique can be taught to students of different levels. Good students learn it quickly and should study the full range of operators of the elementary functions (sin, | |, cos, exp, L*n*, *ax*, *x* + *b*, etc.). For students of lesser ability, it may be better to study only the subset of operators of the most elementary functions ($x+b$, ax, $|x|$). This, nevertheless, is enough to help them understand functions usually found in a physics course.

Although the general method is necessary for functions of the form $f(x)+g(x)$, $f(x)\cdot g(x)$, and $f(x)/g(x)$ (for which it is sometimes even necessary to use l 'Hopital's Rule), the method proposed here covers those functions which result from a composition of elementary functions. It can be learned before the notions of limit, derivatives, etc., have been taught.

There are hand-held calculators now with graphing capabilities and computer packages which graph all the functions mentioned in this article (and many more). Moreover, they may influence the way graphing of functions is taught. The importance of calculating extreme points of a function, complicated limits, etc. will tend to decrease because the machine can do this. In science, the ability to relate formulas to graphs and their interpretation in the particular science is very important. Some examples of situations where this ability is needed are to understand why in $a \cdot \sin(x)$ the a means amplitude of the wave or what is meant by exponential decay or how $\exp(-(x-u)^2)$ and $exp(-(x)^2)$ are related. The comprehension of these ideas needs the previous development of good graphing skills. The method presented in this article helps students to develop such skills. The use of computer packages and hand-held calculators with graphing capabilities for teaching applications should also help students. The development of software which incorporates graphical operators will be very useful for this task. In such kind of software, the operators should be like tools to modify shapes on the plane. Graphs of functions should be seen as particular shapes. The other part should be oriented to discover the name of the modified graph.

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